EE 221: Signal Analysis & Systems
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Fourier Transform: Practice Problems

Q1. Determine the Fourier transform $X(\omega)$ for $x(t) = 10\,\text{sinc}^2\left(\frac{5\,(t+1)}{\pi}\right)$.

Q1. Solution. From Fourier transform tables

$$x(t) = \Delta(t) \Leftrightarrow X(\omega) = \text{sinc}^2\left(\frac{\omega}{2\pi}\right)$$

Hence, using properties of Fourier transform, we get,

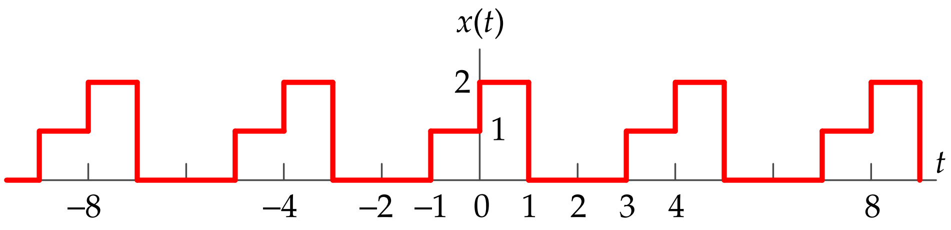
$$X(t) = \text{sinc}^2\left(\frac{t}{2\pi}\right) \Leftrightarrow 2\pi\, x(-\omega) = 2\pi\, \Delta(-\omega)$$

$$\text{sinc}^2\left(10\frac{t}{2\pi}\right) \Leftrightarrow \frac{1}{|10|} 2\pi\, \Delta\left(\frac{1}{10} \times \omega\right)$$

$$10\,\text{sinc}^2\left(\frac{5\,t}{\pi}\right) \Leftrightarrow \frac{10}{10} 2\pi\, \Delta\left(\frac{\omega}{10}\right)$$

$$10\,\text{sinc}^2\left(\frac{5\,(t+1)}{\pi}\right) \Leftrightarrow 2\pi\, \Delta\left(\frac{\omega}{10}\right) e^{j\omega}$$

Q2. For the following signal $x(t)$, determine the complex exponential Fourier series coefficients.



Q1. Solution. Instead of integration, which takes some time, we can simply use Fourier series properties (similar to Fourier transform properties) to evaluate the complex exponential Fourier series coefficients.

NOTE: We **cannot** use Fourier transform properties to evaluate the trigonometric Fourier series coefficients. If you want to use properties, calculate the complex exponential Fourier series, then convert it to the trigonometric form.

From the diagram, we can write,

$$x(t) = \text{rep}_{T_0=4}\{1\,\text{rect}(t+0.5) + 2\,\text{rect}(t-0.5)\}$$

$$x(t) = \text{rep}_4\{\text{rect}(t+0.5)\} + \text{rep}_4\{2\,\text{rect}(t-0.5)\} = x_1(t) + x_2(t)$$

We already know that for $x(t) = \text{rep}_{T_0}\left\{A\,\text{rect}\left(\frac{t}{\tau}\right)\right\}$, the complex exponential Fourier series coefficients are,

$$\alpha_n = \frac{A\tau}{T_0}\text{sinc}\left(\frac{n\omega_0\tau}{2\pi}\right) = \frac{A\tau}{T_0}\text{sinc}\left(\frac{n\tau}{T_0}\right)$$

Using the time-shifting property for $x(t) = \text{rep}_{T_0}\left\{A\,\text{rect}\left(\frac{t\pm t_0}{\tau}\right)\right\}$, the complex exponential Fourier series coefficients become,

$$\alpha_n = \frac{A\tau}{T_0}\text{sinc}\left(\frac{n\omega_0\tau}{2\pi}\right)e^{\pm jn\omega_0 t_0} = \frac{A\tau}{T_0}\text{sinc}\left(\frac{n\tau}{T_0}\right)e^{\pm jn\omega_0 t_0}$$

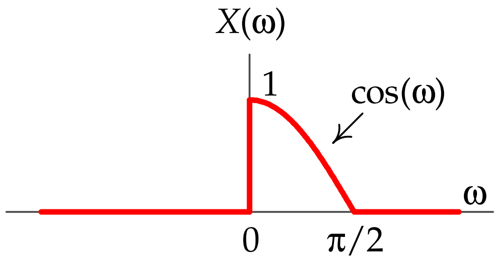
Now, using the superposition property, we have,

$$\alpha_n = \alpha_{n,1} + \alpha_{n,2}$$

$$\alpha_n = \frac{1\times 1}{4}\text{sinc}\left(\frac{n\times 1}{4}\right)e^{+jn\frac{2\pi}{4}0.5} + \frac{2\times 1}{4}\text{sinc}\left(\frac{n\times 1}{4}\right)e^{-jn\frac{2\pi}{4}0.5}$$

$$\alpha_n = \frac{1}{4}\text{sinc}\left(\frac{n}{4}\right)e^{+j\frac{n\pi}{4}} + \frac{1}{2}\text{sinc}\left(\frac{n}{4}\right)e^{-j\frac{n\pi}{4}}$$

Q3. Determine the inverse Fourier transform of $X(\omega)$ shown below.



Q3. Solution. Using the inverse Fourier transform integral

$$x(t) = \mathcal{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{0}^{\frac{\pi}{2}} \cos(\omega)\, e^{j\omega t}d\omega$$

Using Euler's identity,

$$x(t) = \frac{1}{2\pi}\int_{0}^{\frac{\pi}{2}} \left[\frac{e^{j\omega} + e^{-j\omega}}{2}\right] e^{j\omega t}d\omega$$

$$x(t) = \frac{1}{4\pi}\int_{0}^{\frac{\pi}{2}} e^{j\omega}\, e^{j\omega t}d\omega + \frac{1}{4\pi}\int_{0}^{\frac{\pi}{2}} e^{-j\omega}\, e^{j\omega t}d\omega$$

$$x(t) = \frac{1}{4\pi}\int_{0}^{\frac{\pi}{2}} e^{j(t+1)\omega}d\omega + \frac{1}{4\pi}\int_{0}^{\frac{\pi}{2}} e^{j(t-1)\omega}d\omega$$

$$x(t) = \frac{1}{4\pi}\left[\frac{e^{j(t+1)\omega}}{j(t+1)}\right]_0^{\frac{\pi}{2}} + \frac{1}{4\pi}\left[\frac{e^{j(t-1)\omega}}{j(t-1)}\right]_0^{\frac{\pi}{2}}$$

$$x(t) = \frac{1}{4\pi}\left[\frac{e^{j(t+1)\frac{\pi}{2}} - e^0}{j(t+1)}\right] + \frac{1}{4\pi}\left[\frac{e^{j(t-1)\frac{\pi}{2}} - e^0}{j(t-1)}\right]$$

$$x(t) = \frac{1}{4\pi}\left(\frac{e^{j\frac{\pi t}{2}}e^{j\frac{\pi}{2}} - 1}{j(t+1)} + \frac{e^{j\frac{\pi t}{2}}e^{-j\frac{\pi}{2}} - 1}{j(t-1)}\right)$$

$$x(t) = \frac{1}{4\pi}\left(\frac{je^{j\frac{\pi t}{2}} - 1}{j(t+1)} + \frac{-je^{j\frac{\pi t}{2}} - 1}{j(t-1)}\right)$$

$$x(t) = \frac{1}{4\pi}\left(\frac{e^{j\frac{\pi t}{2}} + j}{(t+1)} + \frac{-e^{j\frac{\pi t}{2}} + j}{(t-1)}\right)$$

$$x(t) = \frac{1}{4\pi}\left(\frac{(t-1)e^{j\frac{\pi t}{2}} + j(t-1) - (t+1)e^{j\frac{\pi t}{2}} + j(t+1)}{t^2 - 1^2}\right)$$

$$x(t) = \frac{1}{4\pi}\left(\frac{2jt - 2e^{j\frac{\pi t}{2}}}{t^2 - 1}\right)$$

$$x(t) = \frac{1}{2\pi}\left(\frac{e^{j\frac{\pi t}{2}} - jt}{1 - t^2}\right)$$

Notice that the signal $x(t)$ is a complex-valued signal, which was expected because the magnitude spectrum density does not exhibit even symmetry.